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A NOTE ON FINITE CRACK CROSSING NORMALLY AN INTERFACE WITH LOGARITHMIC SINGULARITY AT THE INTERFACE

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Abstract—When a finite crack crosses normally an interface, the singularity at the intersection with the interface may be only of logarithmic type if the elastic constants are at a certain relation to each other as in the case of two bonded quarterplanes and, in addition, the ratio of the crack lengths is related to the elastic constants as shown in this note.

INTRODUCTION

Consider the plane elastostatic problem of a finite crack, perpendicular to and going through the interface of two bonded elastic materials, characterized by the elastic constants (μ_1, ν_1) and (μ_2, ν_2), where μ_i are the shear moduli and ν_i the Poisson ratios $(i = 1, 2)$ (Fig. 1.). Let b_1, b_2 be the crack lengths in either material and the self-equilibrating surface tractions $-p_1(r)$ and $-p_2(r)$ be the only external loads.

The singular behavior of the solution at the interface is similar to that at the apex of two edge-bonded orthogonal wedges. The asymptotic behavior of this latter problem has been studied by Bogy[1] and it is has been found that for

$$\alpha = 2\beta \quad \text{and} \quad p_1(0)/p_2(0) \neq (1+2\beta)/(1-2\beta) \tag{1a,b}$$

the stress field at the apex presents a logarithmic singularity due to the normal tractions exerted on the free sides of the wedges. In eqns (1a,b) α and β are the composite parameters proposed by Dunders[2]. Equation (1a) may be written in terms of μ_i and ν_i as

$$\nu_1/\nu_2 = \mu_1/\mu_2. \tag{2}$$

The problem of a finite crack crossing normally an interface has been studied by Erdogan and Biricikoglu[3]. The unknown functions f_1, f_2 were defined by

$$f_1(r) = -2\frac{\partial}{\partial r}u_{1\theta}(r, \pi - 0)$$

$$f_2(r) = 2\frac{\partial}{\partial r}u_{2\theta}(r, +0). \tag{3a,b}$$

Assuming a power type singularity of the gradient of the displacement field near the intersection with the interface

$$f_j(s) = \frac{g_j(s)}{s^{\alpha}(b_j - s)^{1/2}}, \quad j = 1,2 \tag{4}$$

and studying the asymptotic behavior of the integral equations for $s \to 0$, it has been determined in[3] that the value of the exponent α is a solution of a non-linear equation, for any pair of bonded materials. Furthermore, the expressions for the stresses $\tau_{\theta\theta}$, $\tau_{r\theta}$ were given and by considering the asymptotic behavior of these expressions for $s \to 0$ the corresponding stress intensity factors k_r, k_θ were determined, expressed in terms of the value of the regular part $g_j(s)$ of $f_j(s)$ for $s = 0$ (eqns (38a,b) of Ref.[3]).

Here the same problem as in[3] is studied, but the singular behavior of $f_j(s)$ near the

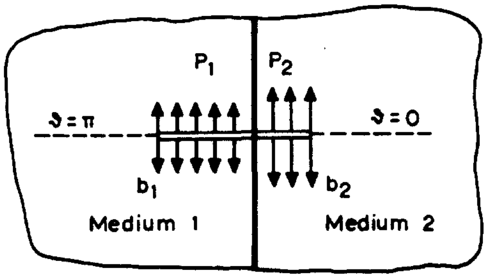

Fig. 1. Crack crossing normally an interface. Medium 1 (μ_1, ν_1), Medium 2 (μ_2, ν_2).

intersection with the interface is assumed to be of logarithmic type: $f_j(s) \sim \ln s/b_j$, $j = 1,2$. On the other hand, the wellknown power singularity $(b_j - s)^{-1/2}$ characterizes the asymptotic behavior at the crack tips away from the interface. Consequently, the density functions $f_j(s)$ are given not anymore by eqn (4), but by the relation

$$f_j(s) = -(b_j - s)^{-3/2} \ln \frac{s}{b_j} g_j(s), \quad j = 1,2 \tag{5}$$

where $g_j(s)$, $0 \le s \le b_j$, $j = 1,2$ satisfy a Hölder condition.

We rewrite from [3] the integral equations reduced to the interval [0, 1] by means of the variable transformation $t_j = s/b_j$

$$\int_0^1 \left[\frac{1}{t-x} + \frac{1}{t+x}\left(c_{11} + \frac{c_{12}x}{t+x} + \frac{c_{13}x^2}{(t+x)^2}\right)\right] w(t)\varphi_1(t)\,\mathrm{d}t + q\int_0^1 \left(d_{11} + \frac{d_{12}x}{x+tq}\right)\frac{w(t)}{x+tq}\varphi_2(t)\,\mathrm{d}t = -\pi\frac{1+\kappa_1}{2\mu_1}p_1 \tag{6a}$$

$$\int_0^1 \left(d_{21} + \frac{d_{22}qx}{qx+t}\right)\frac{w(t)}{qx+t}\varphi_1(t)\,\mathrm{d}t + \int_0^1 \left[\frac{1}{t-x} + \frac{1}{t+x}\left(c_{21} + \frac{c_{22}x}{t+x} + \frac{c_{23}x^2}{(t+x)^2}\right)\right] w(t)\varphi_2(t)\,\mathrm{d}t = -\pi\frac{1+\kappa_2}{2\mu_2}p_2 \tag{6b}$$

and the condition for single—valuedness of displacements

$$\int_0^1 w(t)\varphi_1(t)\,\mathrm{d}t = q\int_0^1 w(t)\varphi_2(t)\,\mathrm{d}t \tag{6c}$$

where, instead of eqn (5), we have

$$f_j(s) = w(t)\varphi_j(t) = -(1-t)^{-3/2}\ln t\varphi_j(t) \tag{7}$$

and

$$c_{11} = \frac{1}{2}\left[1 - \frac{m_1(1+\kappa_1)}{m_1+\kappa_2} - \frac{3(1-m_1)}{1+m_1\kappa_2}\right] \tag{8}$$

$$c_{12}=6\frac{1-m_1}{1+m_1\kappa_1},\quad c_{12}=-4\frac{1-m_1}{1+m_1\kappa_1}=-\frac{3}{2}c_{12}$$

$$c_{21}=\frac{1}{2}\left[1-\frac{m_2(1+\kappa_2)}{m_2+\kappa_1}-\frac{3(1-m_2)}{1+m_2\kappa_2}\right]$$

$$c_{22}=6\frac{1-m_2}{1+m_2\kappa_2},\quad c_{23}=-4\frac{1-m_2}{1+m_2\kappa_2}=-\frac{2}{3}c_{22} \tag{8}$$

$$d_{11}=\frac{1+\kappa_1}{2}\left(\frac{3}{m_2+\kappa_1}-\frac{1}{1+m_2\kappa_2}\right),\quad d_{12}=(1+\kappa_1)\left(\frac{1}{1+m_2\kappa_2}-\frac{2}{m_2+\kappa_1}\right)$$

$$d_{21}=\frac{1+\kappa_2}{2}\left(\frac{3}{m_1+\kappa_2}-\frac{1}{1+m_1\kappa_1}\right),\quad d_{22}=(1+\kappa_2)\left(\frac{1}{1+m_1\kappa_1}-\frac{1}{m_1+\kappa_2}\right)$$

with

$$m_1=\mu_2/\mu_1,\quad m_2=\mu_1/\mu_2,\quad q=b_2/b_1 \tag{9}$$

and

$$\kappa_j=\left\{\begin{array}{ll}3-4\nu_j & \text{'for plain strain}\\ \dfrac{3-\nu_j}{1+\nu_j} & \text{for generalized plane stress}\end{array}\right\}\ j=1,2.$$

We next come to consider the asymptotic behavior of the integral equations, to which the problem was reduced. We restrict ourselves to the behavior at the interface, since the asymptotic study at the crack tips away from the interface would lead to known power singularity of order $-1/2$, already incorporated into the weight function in eqn (7). For $t\to 0$, employing Gakhov's equations (8.30) and (8.25) (Ref. [4]), for x outside the integration interval (i.e. for the terms in the integral eqns (6) presenting the denominator $(t+x)$) and eqns (8.31) and (8.25) of Ref. [4] for x inside the integration interval (i.e. terms in eqns (6) with the denominator $(t-x)$), we obtain the following asymptotic expressions of eqns (6)

$$-\frac{1+\kappa_1}{2\mu_1}\pi p_1=-\frac{1}{2}[(1+c_{11})\varphi_1(0)+d_{11}\varphi_2(0)]\ln^2 x$$
$$+\frac{1}{2}[(2c_{12}+c_{13})\varphi_1(0)+2(d_{12}+d_{11}\ln q)\varphi_2(0)]\ln x+\Phi_1(x) \tag{10a}$$

$$-\frac{1+\kappa_2}{2\mu_2}\pi p_2=-\frac{1}{2}[d_{21}\varphi_1(0)+(1+c_{21})\varphi_2(0)]\ln^2 x$$
$$+\frac{1}{2}[2(d_{22}-d_{21}\ln q)\varphi_1(0)+(2c_{22}+c_{23})\varphi_2(0)]\ln x+\Phi_2(x) \tag{10b}$$

where, $\Phi_j(x)$, $j=1,2$, are analytic at the point $x=0$. From the requirement that the coefficients of $\ln^2 x$, $\ln x$ in eqns (10a,b) vanish at the vicinity of $x=0$, we obtain the following two homogeneous linear systems in $\varphi_1(0)$ and $\varphi_2(0)$

$$\left.\begin{array}{l}(1+c_{11})\varphi_1(0)+d_{11}\varphi_2(0)=0\\ d_{21}\varphi_1(0)+(1+c_{21})\varphi_2(0)=0\end{array}\right\} \tag{11}$$

and

$$\left.\begin{array}{l}(2c_{12}+c_{13})\varphi_1(0)+2(d_{12}+d_{11}\ln q)\varphi_2(0)=0\\ 2(d_{22}-d_{21}\ln q)\varphi_1(0)+(2c_{22}+c_{23})\varphi_2(0)=0\end{array}\right\}. \tag{12}$$

The condition for non-trivial solutions in eqns (11) is

$$(1+c_{11})(1+c_{21}) = d_{11}d_{21} \tag{13}$$

and substituting from eqns (8) it is found that eqn (13) is an identity for μ_j, ν_j, $j = 1,2$. Furthermore, the non-zero solutions of eqns (11) are linearly dependent, that is

$$\varphi_2(0) = -\varphi_1(0)(1+c_{11})/d_{11}. \tag{14}$$

Because of eqn (14) and since $\varphi_j(0) \neq 0$, $j = 1,2$, eqns (12) may be written

$$\left.\begin{aligned}(2c_{12}+c_{13})d_{11} - 2(d_{12}+d_{11}\ln q)(1+c_{11}) = 0\\ 2(d_{22}-d_{21}\ln q)d_{11} - (2c_{22}+c_{23})(1+c_{11}) = 0\end{aligned}\right\}. \tag{15}$$

Equations (15) must both hold in order that the assumption for the logarithmic singularity of $f_j(s)$ at the interface be valid. These equations have two implications:

(i) By eliminating $\ln q$ we find

$$(2c_{12}+c_{13})\frac{d_{11}}{1+c_{11}} + (2c_{22}+c_{23})\frac{1+c_{11}}{d_{21}} = 2d_{22}\frac{d_{11}}{d_{21}} + 2d_{12} \tag{16}$$

(ii)

$$\ln q = \frac{2c_{12}+c_{13}}{2(1+c_{11})} - \frac{d_{12}}{d_{11}}. \tag{17}$$

Equation (16) states the relation that must hold between the elastic constants, therefore it is expected to be equivalent to Bogy's condition stated by eqn (2). In verifying it, it is algebraically more convenient to proceed from eqn (2) to (16). Actually, because of eqn (2)

$$c_{11} = c_{21},\ d_{11} = d_{21},\ 1+c_{11} = d_{11} \tag{18}$$

and eqn (16) takes the form

$$c_{12}+c_{22}+(c_{13}+c_{23})/2 = d_{12}+d_{22} \tag{19}$$

which is, because of eqn (2), satisfied.

On the other hand, eqn (17) shows that the ratio b_2/b_1 of the crack lengths must be at a certain relation to the elastic constants. This relation, because of eqn (16) (or, equivalently, of eqns (18)) takes the simpler form

$$\ln q = (c_{12}+c_{13}/2 - d_{12})/d_{11}. \tag{20}$$

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